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Maximal partial spreads and the modular n-queen problem III

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Abstract

Maximal partial spreads in PG$(3,q)$ $q = p^k$, p odd prime and $q \geqslant 7$, are constructed for any integer n in the interval $(q^2+1)/2+6 \leqslant n \leqslant (5q^2+4q-1)/8$ in the case $q+1 \equiv 0, \pm 2, \pm 4, \pm 6, \pm 10, 12 \pmod{24}$. In all these cases, maximal partial spreads of the size $(q^2+1)/2+n$ have also been constructed for some small values of the integer n. These values depend on q and are mainly $n=3$ and $n=4$. Combining these results with previous results of the author and with that of others we can conclude that there exist maximal partial spreads in PG$(3,q)$, $q = p^k$ where p is an odd prime and $q \geqslant 7$, of size n for any integer n in the interval $(q^2+1)/2+6 \leqslant n \leqslant q^2-q+2$. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

A *maximal partial spread* in PG$(3,q)$ is a set of lines $S = \{\ell_1, \ell_2, \ldots, \ell_k\}$ satisfying the following two conditions:

(i) $\ell_i \cap \ell_j = \emptyset$ for $i \neq j$;
(ii) to any line $\ell \in \mathrm{PG}(3,q) \backslash S$ there is a line $\ell_i \in S$, such that $\ell \cap \ell_i \neq \emptyset$.

Maximal partial spreads were first studied by Dale Mesner [14] in 1967. There has been a search for maximal partial spreads for over 30 years but there is still no final solution of the problem of the determination of the possible number of lines k of a maximal partial spread. Only for small values of q, namely $q = 3, 4, 5$, the possible values of k are known, see [13,12].

The search for maximal partial spreads described below will in some way contain a part of a possible complete solution. We prove that for any integer n in the interval

$$\frac{q^2+1}{2} + 6 \leqslant n \leqslant q^2 - q + 2$$

there is a maximal partial spread of size n in PG$(3,q)$, q odd, $q \geqslant 7$.

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The case $n=q^2-q+2$ was first proved by Bruen [3]. The remaining cases follow, as will be shown below, from the method developed by the author in [10]. Some of the maximal partial spreads we have been able to construct by our methods were already found by others such as Beutelspacher [1] and Ebert [6].

For certain values of q the above result can be improved. We have also found maximal partial spreads of the following sizes:

$$\frac{q^2+1}{2}+n \quad \text{for } n=3,4 \text{ and } 5 \quad \text{if } q+1\equiv \pm 2 \pmod{6};$$

$$\frac{q^2+1}{2}+n \quad \text{for } n=4 \quad \text{if } q+1\equiv 0 \pmod{6},\ q\geqslant 17;$$

$$\frac{q^2+1}{2}+n \quad \text{for } n=1,2,3,4 \text{ and } 5 \quad \text{if } q+1\equiv \pm 2 \pmod{12};$$

$$\frac{q^2+1}{2}+n \quad \text{for } n=3,4 \text{ and } 5 \quad \text{if } q+1\equiv \pm 4 \pmod{12};$$

$$\frac{q^2+1}{2}+n \quad \text{for } n=3 \text{ and } 5 \quad \text{if } q=11.$$

The interval $(5q^2+4q-1)/8\leqslant n\leqslant q^2-q+2$ was considered, for any odd $q\geqslant 7$, in [10]. The interval $(q^2+1)/2+3\leqslant n\leqslant (5q^2+4q+7)/8$ in case $q+1\equiv \pm 8 \pmod{24}$ was considered in [11]. The proof of the remaining cases are given below and are divided into five sections: In Section 3 we consider the case $q+1\equiv 0 \pmod{12}$, $q\geqslant 23$, in Section 4 the case $q+1=12$, in Section 5 the case $q+1\equiv 6 \pmod{12}$, in Section 6 the case $q+1\equiv \pm 2 \pmod{12}$ and finally in Section 7 the case $q+1\equiv \pm 4 \pmod{24}$.

In the interval $q^2-q+3\leqslant n\leqslant q^2$ there are no known maximal partial spreads. For $n\leqslant (q^2+3)/2$ there are only three maximal partial spreads known. They were found by the author in a computer search [9]. In that search maximal partial spreads of the sizes 23, 24 and 25 in $\mathrm{PG}(3,7)$ were found.

Let us also mention two non-existence results. Glynn proved in 1981 [7] that the number of lines k of any maximal partial spread in $\mathrm{PG}(3,q)$ must be greater than $2k-1$. Let δ denote the *deficiency*, that is, the number $\delta=q^2+1-|S|$, for a maximal partial spread S in $\mathrm{PG}(3,q)$. It follows from a famous result of Blokhuis on blocking sets [2] that if q is a prime then $\delta\geqslant (q+3)/2$.

An easy way to construct a maximal partial spread is to start with a regular spread S. Consider one line ℓ of S and two reguli R_1 and R_2 of S both containing the line ℓ. By taking away from S the lines of the reguli R_1 and R_2 and completing with lines from the two opposite reguli of R_1 and R_2 one will get a maximal partial spread of size less than q^2+1.

Our construction is similar. We consider one line ℓ of the regular spread S and a special set of mutually disjoint reguli $R_0, R_1, \ldots, R_{m-1}$, $m=(q+1)/2$, not containing ℓ. The set of lines that only intersect ℓ and the lines of these reguli can be described as bishops on a modular chessboard of size $(q+1)\times(q+1)$. These lines meet if and only if the corresponding bishops attack. It will also be easy to describe the intersection of these bishop lines and the lines of the reguli $R_0, R_1, \ldots, R_{m-1}$ and their opposite reguli. This procedure will be described more precisely in Section 2. There we also give a formula for calculating the number of lines of the maximal partial spreads obtained in this way.

A completely different way to construct maximal partial spreads with a small number of lines was given by Bruen and Hirschfeld [4]. They used methods from algebraic geometry.

2. Preliminaries

For a general introduction to finite geometry, see e.g. [5]. Below some of the results of [10] are reviewed. Hence for details, consult [10].

We will consider $\mathrm{PG}(3,q)$ as the direct product

$$\mathrm{PG}(3,q) = \mathrm{GF}(q^2) \times \mathrm{GF}(q^2),$$

where the finite field $\mathrm{GF}(q^2)$ is considered as a vector space of dimension two over the finite field $\mathrm{GF}(q)$. Lines of $\mathrm{PG}(3,q)$ will be the 2-dimensional subspace of this direct product. Two such lines are skew if and only if their intersection is the zero vector.

Let ℓ_∞ denote the line

$$\ell_\infty = \{0\} \times \mathrm{GF}(q^2).$$

It was observed in [10] that the lines of $\mathrm{PG}(3,q)$, skew to the line ℓ_∞, are the following subspaces of $\mathrm{GF}(q^2) \times \mathrm{GF}(q^2)$:

$$[a,b] = \{(x, ax + bx^q) \mid x \in \mathrm{GF}(q^2)\},$$

where $a, b \in \mathrm{GF}(q^2)$.

All constructions described below originate from the regular spread S of $\mathrm{PG}(3,q)$. We will describe S as the set of lines

$$S = \{\ell_\infty\} \cup \{[a,0] \mid a \in \mathrm{GF}(q^2)\}.$$

Let η be a primitive element of $\mathrm{GF}(q^2)$. Let H denote the following subgroup of the multiplicative group of $\mathrm{GF}(q^2)$:

$$H = \{\eta^{(q-1)}, \eta^{2(q-1)}, \ldots, \eta^{(q+1)(q-1)} = 1\}.$$

Let $a_0, a_1, \ldots, a_{q-2}$ be a set of coset representatives of H in the multiplicative group of $\mathrm{GF}(q^2)$. The following sets of lines will constitute a family of $q-1$ mutually disjoint reguli of the regular spread S:

$$R_i = [a_iH, 0] = \{[a_ih, 0] \mid h \in H\}, \quad i = 0, 1, \ldots, q-2.$$

Their opposite reguli are the following sets of lines:

$$R_i^{\mathrm{opp}} = [0, a_iH] = \{[0, a_ih] \mid h \in H\}, \quad i = 0, 1, \ldots, q-2.$$

The set of lines

$$B = \{[h,k] \mid h,k \in H\}$$

is fundamental in our investigation. Let $R_{[h,k]}$ denote the set of reguli, from the above family of reguli, that the line $[h,k]$ of B meet, i.e.

$$R_i \in R_{[h,k]} \quad \text{if and only if there is } \ell \in R_i \text{ such that } [h,k] \cap \ell \neq \emptyset.$$

The following theorem was proved in [10]:

Theorem 2.1. *Assume q is an odd prime power and $q \geqslant 7$.*

(i) *For any $h,k \in H$, $R_{[h,k]} = R_{[1,1]}$.*
(ii) *If the only lines of S that the line ℓ intersects are lines from the set of reguli in $R_{[1,1]}$ then $\ell = [h,k]$ for some $h,k \in H$.*

In [10] we also found some simple criteria for the intersection of the lines $[h,k]$, $h,k \in H$ and the lines from the reguli of $R_{[h,k]}$. In order to describe these criteria as concisely as possible we have to switch to a new notation. Here (i,j) will denote the following line:

$$(i,j) = [\eta^{i(q-1)}, \eta^{-j(q-1)}].$$

Lemma 2.1. *(i,j) and (i',j') intersect if and only if either $i+j \equiv i'+j' \pmod{q+1}$ or $i-j \equiv i'-j' \pmod{q+1}$.*

We also proved that it was possible to number the lines of the reguli of $R_{[1,1]}$ and to renumber these reguli such that

$$R_{[1,1]} = \{R_0, R_1, \ldots, R_{m-1}\}, \quad m = (q+1)/2$$

and such that the following lemma will be true:

Lemma 2.2. (i) *The line (i,j) intersects the lines i and $i+e \pmod{q+1}$ of R_e for $e = 0,1,\ldots,m-1$.*

(ii) *The line (i,j) intersects the lines j and $j+e \pmod{q+1}$ of R_e^{opp} for $e = 0,1,\ldots,m-1$.*

The maximal partial spreads that we are going to construct will consist of the lines from the set

$$\{\ell_\infty\} \cup \{\ell \in R_i \mid i = m, m+1, \ldots, q-2\}$$

completed with lines from the set B, described above, and lines from either R_k or R_k^{opp} for $k = 0,1,\ldots,m-1$. If it is impossible to find further skew lines from the set B or from the reguli R_k, respectively, R_k^{opp}, $k = 0,1,\ldots,m-1$, then, according to Theorem 2.1, the partial spread will be maximal. Lemmas 2.1 and 2.2 will make it possible to calculate the number of lines of the maximal partial spread. The most convenient way to perform these calculations is to consider the elements of B as bishops on a modular chessboard of size $(q+1)\times(q+1)$.

A *modular chessboard* of size $(q+1)\times(q+1)$ is a set of squares (i,j), $i=0,1,\ldots,q$ and $j=0,1,\ldots,q$. The chessboard has rows and columns. The *row number* i will be the set

$$(i,*)=\{(i,k)\mid k=0,1,\ldots,q\},\quad i=0,1,2,\ldots,q$$

and the *column number* j is the set

$$(*,j)=\{(k,j)\mid k=0,1,\ldots,q\},\quad j=0,1,2,\ldots,q.$$

There are two kinds of diagonals. The *diagonal* d_k is the set

$$d_k=\{(i,j)\mid i+j\equiv k\,(\mathrm{mod}\,(q+1)),\ i=0,1,\ldots,q\},\quad k=0,1,2,\ldots,q$$

and the *bidiagonal* bd_k is the set

$$bd_k=\{(i,j)\mid i-j\equiv k\,(\mathrm{mod}\,(q+1)),\ i=0,1,\ldots,q\},\quad k=0,1,2,\ldots,q.$$

By Lemma 2.1 two lines $[h,k]$, $h,k\in H$ intersect if and only if the two bishops are on the same diagonal or bidiagonal of the chessboard.

The lines of the regulus R_k and its opposite regulus R_k^{opp} may be considered as the rows and columns respectively, of the modular chessboard. By Lemma 2.2, a bishop at the square (i,j) will intersect the row number i and $i+k$ and the column number j and $j+k$.

We give an example in the case $q=11$. It is required in the proof of our result.

Example. Let $q=11$. We consider the following set of bishops on a chessboard of size 12×12:

$$B=\{(0,0),(1,2),(2,4),(3,6),(4,3),$$
$$(4,9),(6,10),(7,1),(8,3),(9,5),(10,7),(0,10)\}.$$

This is a set of 12 non-attacking bishops. The only rows and columns that are empty are rows 5 and 11 and the columns 8 and 11. The only lines from the reguli $R_0,R_1,R_2,\ldots,R_5$ and their opposite reguli that do not intersect any of the bishops are lines 5 and 11 from R_0 (or 8 and 11 from R_0^{opp}) and line number 11 from R_3^{opp}. Hence by completing with these lines we obtain a maximal partial spread which contains 64 lines.

All our constructions below will start with a set B of $q+1$ non-attacking bishops on a modular chessboard of size $(q+1)\times(q+1)$. A set of $q+1$ non-attacking bishops is said to be a *complete set* of bishops.

If a line of a regulus R_k for $k=0,1,\ldots,m-1$ or its opposite regulus does not intersect any of the bishops of B then this line is a *free line* of the regulus.

With this terminology we have the following theorem.

Theorem 2.2. *Let B be a complete set of bishops with m_k free lines of the regulus R_k, $k=0,1,\dots,m-1$, and m_k^{opp} free lines of the opposite regulus R_k^{opp}, $k=0,1,2,\dots,$ $m-1$. Then there is a maximal partial spread of the size*

$$\frac{q^2+1}{2}+\sum_{k=0}^{m-1} n_k,$$

where in case $m_k m_k^{\mathrm{opp}} \neq 0$ we can choose n_k to be either m_k or m_k^{opp}. In case $m_k m_k^{\mathrm{opp}}=0$ we have to choose $n_k=\max(m_k, m_k^{\mathrm{opp}})$.

Maximal partial spreads constructed in this way will be said to be obtained from the set B by *completing with free lines*. The number of lines of the maximal partial spread will, besides the possibilities of 'switches' from free lines of R_k to free lines of R_k^{opp}, depend on the chosen set of bishops.

We now discuss two simple methods of transforming complete sets of bishops into other complete sets of bishops:

If we move a bishop at position (x,y) to the position $(x+m, y+m)$, $m=(q+1)/2$, we obtain a new complete set of bishops from any given complete set of bishops. The position $(x+m, y+m)$ will be called the *complementary position* of the position (x,y).

As $q+1$ is assumed to be an even number, the set of bishops can be divided into two categories, black and white bishops. A bishop in position (x,y) is a *white* bishop if $x+y$ is an even number. If $x+y$ is odd, the bishop is a *black* bishop.

If a bishop is moved from position (x,y) to the position $(x+t,y)$, where $t>0$, we will say that the bishop has been *moved t steps up* the chessboard. If $t<0$ then the bishop has been moved and $-t$ steps *down* the chessboard.

If all black bishops of a complete set of bishops are moved the same even number of steps up (or down) the chessboard, then we get another complete set of bishops.

We now give the main construction idea. We start with a complete set of bishops which occupies the rows $0,1,\dots,m-1$ and with each of these rows containing one black and one white bishop. Further, each column will contain one bishop. We then move all black bishops $2k$ steps up the chessboard and delete two bishops in different rows but in rows containing two bishops.

If the two deleted bishops were at *column distance* $d \leqslant m-1$, which means that they are at the columns number i and $i+d \,(\mathrm{mod}\,(q+1))$ for some integer i, and if they are not replaceable after completing with free lines then, by Theorem 2.2, the size of the maximal partial spread will be

$$\frac{q^2+1}{2}+(1+2+\cdots+(m-2k-1))+d+1$$

if free lines are choosen from the regulus R_d^{opp} and the remaining free lines from the reguli R_k, for $k \neq d$. If we choose free lines from R_0^{opp} and R_d^{opp} and the remaining free lines from the reguli R_k for $k \neq 0$ and $k \neq d$ then the number of lines will be

$$\frac{q^2+1}{2}+(1+2+\cdots+(m-2k-2))+d+2.$$

By choosing suitable values of d we get maximal partial spreads of desired sizes. The problem is that a set of bishops like the set we just described never exists. In the following sections we will see that we can find complete set of bishops very close to this ideal set of bishops.

3. The case $q+1 \equiv 0 \pmod{12}$, $q \geqslant 23$

The proof of the result in this section is given in three subsections.

3.1. *The interval* $(q^2+1)/2+4 \leqslant n \leqslant (q^2+1)/2+13$, $n \neq (q^2+1)/2+5$.

In this subsection we will use a set of bishops B' found by Monsky [16].

Let $M=(q+1)/12$. B' is the following set:

$$\begin{aligned} B' = &\{(t,3t) \mid t=0,1,\ldots,3M-2\} \cup \{(3M-1,9M)\} \\ &\cup \{(3M+t,9M+3+3t) \mid t=0,1,\ldots,M-3\} \cup \{(4M-1,12M-1)\} \\ &\cup \{(4M+t,2+3t) \mid t=0,1,\ldots,2M-2\} \cup \{(6M-1,6M-2)\} \\ &\cup \{(9M+1+t,3M+1+3t) \mid t=0,1,\ldots,M-2\} \cup \{(10M,6M-1)\} \\ &\cup \{(10M+1+t,6M+2+3t) \mid t=0,1,\ldots,2M-2\}. \end{aligned}$$

Exactly two of the rows and two of the columns of the chessboard are empty. The empty rows are $4M-2$ and $9M-1$ and the empty columns are $9M-3$ and -3. The above set of bishops is not complete. We adjoin to B' two bishops at the positions $(8M-1,4M-3)$ and $(11M-2,5M-3)$ and get

$$B = B' \cup \{(8M-1,4M-3),\ (11M-2,5M-3)\}.$$

The empty rows and columns will still be empty. By completing the set of bishops with free lines we obtain a maximal partial spread of size $(q^2+1)/2+4$.

Next, we move the bishop in the position $(11M-2,9M-1)$ to its complementary position. No new rows of the chessboard will be empty, but there is one new empty column, the column number $9M-1$. By completing with free lines we obtain maximal partial spreads of the sizes $(q^2+1)2+6$ and $(q^2+1)/2+7$.

If we start with the set B again and move the bishop at position $(0,0)$ to its complementary position $(6M,6M)$ then we get a maximal partial spread of size $(q^2+1)/2+9$.

If we start with B, move the bishop in the position $(11M-2,9M-1)$ to its complementary position and delete the bishop in the position $(2M-1,6M-3)$ we obtain a chessboard with three empty rows, the rows number $2M-1$, $4M-2$ and $9M-1$, and four empty columns, the columns $9M-3$, $6M-3$, -3 and $9M-1$. By using this chessboard we will obtain maximal partial spreads of the sizes $(q^2+1)/2+10$ and $(q^2+1)/2+11$.

Finally, if we start the set B again, by moving the two bishops at the positions $(0,0)$ and $(11M-2, 9M-1)$ to their complementary positions we will obtain maximal partial spreads of the sizes $(q^2+1)/2+12$ and $(q^2+1)/2+13$.

To obtain a maximal partial spread of size $(q^2+1)/2+8$, we move the bishop at position $(6m-1, 6m-2)$ to its complementary position. This chessboard will have three empty rows and three empty columns. We observe that one of the column distances equals one of the row distances and thus we obtain by completing with free lines a maximal partial spread of the size desired.

3.2. *The interval* $(q^2+1)/2+14 \leqslant n \leqslant (q^2+1)/2+47$

Let $m=(q^2+1)/2$. In this case, we consider the following set of bishops:

$$\begin{aligned}
B' = &\{(t,\, 3t) \mid t=0,1,\ldots,m/2-1\}\\
&\cup \{(m/2-1+t,\, 3m/2-1+3t) \mid t=0,1,\ldots,m/2-1\}\\
&\cup \{(m-1+t,\, -5+3t) \mid t=0,1,\ldots,m/2-2\}\\
&\cup \{(3m/2+1+t,\, m/2-5-3t) \mid t=0,1,\ldots,m/6-2\}\\
&\cup \{(10m/6-1+t,\, -3-3t) \mid t=0,1,\ldots,m/6-1\}\\
&\cup \{(11m/6-2+t,\, 3m/2-4-3t) \mid t=0,1,\ldots,m/6-1\}.
\end{aligned}$$

This chessboard will contain five empty rows and two empty columns. The empty rows are the rows -1, -2, $3m/2-2$, $3m/2-1$ and $3m/2$. The empty columns are the columns -2 and $m/2-2$.

As B' contains $q-1$ bishops it is not complete. We will complete it with two bishops. There are two possible ways to perform this completion. We choose one of them such that the above empty columns and rows will remain empty. This complete set of bishops is denoted by B below.

By completing with free lines we obtain maximal partial spreads of the sizes

$$\frac{q^2+1}{2}+n \quad \text{for } n=11,12,14 \text{ and } 15.$$

Next, we move the bishop at position $(0,0)$ to its complementary position (m,m). By this operation the column 0 and row 0 will be empty. Again, completing in the usual way with free lines we get maximal partial spreads of the following sizes:

$$\frac{q^2+1}{2}+n \quad \text{for } n=15,16,17,\ldots,21.$$

We now move all black bishops in the set B two steps down the chessboard. The same set of columns will be empty but now there are seven empty rows. This new complete set of bishops will be denoted B_1.

From the set of bishops B_1 we delete the bishop in position $(m-3,-5)$, and move the bishop in position $(0,0)$ to its complementary position. Some new rows and columns will be empty. If we complete with free lines, the deleted bishops will not be replaceable. The maximal partial spreads obtained will have the following sizes:

$$\frac{q^2+1}{2}+n \quad \text{for } n=26,27,28,\ldots,36.$$

Next, we obtain a new complete set of bishops B_2 by moving all black bishops of the set B_1 two further steps down the chessboard. The same columns as before will be empty, but there are further two empty rows. We move the bishop in position $(m-5,-5)$ to the position $(m-3,-7)$ and the bishop in position $(m-4,-8)$ to the position $(m-6,-6)$. Still the set of bishops is complete. Two new empty columns have appeared they are columns -5 and -8. Completing the partial spread in the usual way with free lines will give us maximal partial spreads of the following sizes:

$$\frac{q^2+1}{2}+n \quad \text{for } n=37,38,\ldots,47.$$

Finally, we make a small change of the originally complete set of bishops B. We move the bishop at position $(-3,m-1)$ to the position $(m-2,-2)$. Another two bishops have to be moved in order to get a complete set of bishops, the two bishops added to the set B' of bishops. We perform these movements and move the bishop in position $(0,0)$ to its complementary position. Completing with free lines in the usual way leads to maximal partial spreads of the sizes

$$\frac{q^2+1}{2}+n \quad \text{for } n=20,21,\ldots,29.$$

3.3. *The interval* $(q^2+1)/2+48 \leqslant n \leqslant (5q^2+4q-1)/8$

Let $m=(q+1)/2$. In this section we consider the following set of bishops:

$$\begin{aligned}
B' = {} & \{(t,\, 3t) \mid t=0,1,\ldots,m/2-1\} \\
& \cup \{(m/2-1+t,\, 3m/2-1+3t) \mid t=0,1,\ldots,m/2-1\} \\
& \cup \{(t,\, m+1+3t) \mid t=0,1,\ldots,m/2-1\} \\
& \cup \{(m/2-3+t,\, m/2-2+3t) \mid t=0,1,\ldots,m/6\} \\
& \cup \{(4m/6-3+t,\, m+2+3t) \mid t=0,1,\ldots,m/6-2\} \\
& \cup \{(5m/6-5+t,\, 3m/2+3t) \mid t=0,1,\ldots,m/6-3\}.
\end{aligned}$$

This chessboard will contain $m+1$ empty rows, the rows $m-1$, m, $m+1,\ldots,q$. There are three empty columns, the columns $m-1$, -3 and -6. To get a complete set of bishops we have to add two bishops. There are two possibilities to perform this. We make the completion in such a way that the empty rows and columns will still be empty. The complete set of bishops obtained will be denoted by B.

We will also consider the following set of bishops:

$$(B' \backslash \{(0,m+1),\, (1,m+4)\}) \cup \{(m-7,-6),\, (m-6,-3)\}.$$

This chessboard will contain the same empty rows as the previous chessboard. The three empty columns will be the columns $m-1$, $m+1$ and $m+4$. We make the set of bishops complete by adding two bishops in such a way that the empty rows and columns will remain empty. This complete set of bishops will be denoted B^*.

If we move all black bishops an even number of steps up the chessboard, the new set of bishops will be complete and the same columns will be empty. For this set of bishops the empty rows will be the rows $m-1+2k-4, m-1+2k-3, \ldots, q$, for some integer k such that $2k \geqslant 4$. For the set of bishops B^*, the empty rows will be the rows $m-1+2k-2$, $m-1+2k-1, \ldots, q$.

Some rows will contain two bishops. If we delete one such bishop and fill up with free lines, the deleted bishop will not be replaceable, as shown in the example below. In this way, we obtain maximal partial spreads of size n for all integers n in the prescribed interval. The proof of this is straightforward, but tedious. We give one example of the calculations and leave the rest to the interested reader.

Example. We move all black bishops of the set B^* four steps up the chessboard. We delete one of the bishops in the set

$$\left\{ \left(\frac{4m}{6} + t, 2 + 3t \right) \middle| \ t = 0, 1, \ldots, \frac{m}{3} - 3 \right\}.$$

Now, the chessboard has four empty columns and $m-1$ empty rows. The empty columns will be $m-1$, $m+1$, $m+4$ and one of the columns in the set $\{2, 5, 8, \ldots, m-7\}$. If we use lines from the set of reguli

$$(\{R_0, R_1, \ldots, R_{m-2}\} \backslash \{R_i, R_j\}) \cup \{R_i^{\mathrm{opp}}, R_j^{\mathrm{opp}}\}$$

$$i \in \{2, 3\} \quad j \in \{6, 7, 8, 9, \ldots, m-3\} \backslash \{7, 10, 13\},$$

then the deleted bishop cannot be replaced. The number of free lines of the regulus R_k will be $m-1-k$. There will be only one free line of R_i^{opp} and one free line of R_j^{opp}. Hence, the size of the maximal partial spread obtained will be

$$\frac{q^2+1}{2} - 1 + (1 + 2 + 3 + \cdots + (m-1)) + (1 - (m-i)) + (1 - (m-j)).$$

This expression can be simplified to

$$\frac{q^2+1}{2} + (1 + 2 + 3 + \cdots + (m-3)) + i + j - 2.$$

By choosing suitable pairs i and j we can produce maximal partial spreads of size n for all integers n in the interval

$$\frac{5q^2 - 8q + 59}{8} \leqslant n \leqslant \frac{5q^2 - 4q + 23}{8}.$$

We get another interval by using the same sets of bishops as above, but choosing free lines from the regulus R_0^{opp} instead of from the regulus R_0.

4. The case $q = 11$

The example of Section 2 shows the existence of a maximal partial spread of size 64. It remains to construct maximal partial spreads of the sizes $66, 67, \ldots, 80$.

Let B denote the set of bishops of the example of Section 2. If we move the bishop at the position $(0, 0)$ to its complementary position $(6, 6)$, we get maximal partial spreads of the sizes 66 and 67.

If we also move the bishop at position $(10, 7)$ to its complementary position, we get maximal partial spreads of the sizes 69, 70 and 71.

Further, if we also move the bishop at the position $(9, 5)$ to its complementary position we get maximal partial spreads of the sizes 71, 72, 73 and 74. If we also move the bishops at the positions $(8, 3)$ and $(7, 1)$ to their complementary positions, we get maximal partial spreads of the sizes 76, 77, 78, 79, 80 and 81.

If we start with the set B again, and only move the bishop at the position $(10, 7)$ to its complementary position, the size of the maximal partial spread will be 68. If we then also move the three bishops at the positions $(9, 5)$, $(8, 3)$ and $(7, 1)$ to their complementary positions we obtain maximal partial spreads of the sizes 75, 77, 79 and 81.

5. The case $q + 1 \equiv 6 (\mathrm{mod}\, 12)$

In this section we will consider three different sets of bishops. These sets are discussed in three subsections.

5.1. *The sizes $(q^2 + 1)/2 + n$ for $n = 4$, 6 and 7*

Let M be the integer such that $q + 1 = 12M + 6$. The following set of bishops was found by Monsky [16]:

$$
\begin{aligned}
B' = {} & \{(i, 3i) \mid i = 0, 1, \ldots, 2M - 1\} \\
& \cup \{(2M, 6M + 1)\} \\
& \cup \{(2M + 1 + i, 6M + 4 + 3i) \mid i = 0, 1, \ldots, 4M\} \\
& \cup \{(6M + 2, 6M)\} \\
& \cup \{(6M + 4, 6M + 3)\} \\
& \cup \{(6M + 5 + i, 6M + 6 + 3i) \mid i = 0, 1, \ldots, 2M - 2\} \\
& \cup \{(8M + 5, -1)\} \\
& \cup \{(8M + 6 + i, 2 + 3i) \mid i = 0, i, \ldots, 4M - 1\}.
\end{aligned}
$$

We adjoin the bishops at the positions $(4M+2, 4M-2)$ and $(10M+8, 2M+2)$ to B'. This complete set of bishops is denoted by B.

The chess board will have two empty rows and two empty columns, the rows $6M+3$ and $8M+4$ and columns -3 and -4. By using lines from the reguli R_0, R_{2M+1} and the regulus R_1^{opp} we get a maximal partial spread of the size

$$\frac{q^2+1}{2}+4.$$

Next, we move the bishop at the position $(4M+2, 1)$ to its complementary position $(12M+5, 6M+4)$. By this operation a new empty column has appeared, the column number 1. By using free lines from the reguli R_0 or R_0^{opp} and free lines from R_{2M+1}, R_1^{opp}, R_4^{opp} and R_5^{opp} we obtain maximal partial spreads of the sizes

$$\frac{q^2+1}{2}+n \quad \text{for } n=6 \text{ and } 7.$$

5.2. *The interval* $(q^2+1)/2+8 \leqslant n \leqslant (q^2+1)/2+47$

Let $m=(q+1)/2$. Let B' denote the following set of bishops on a modular chessboard of size $(q+1)\times(q+1)$:

$$\begin{aligned} B' = &\left\{ (i, 3i) \,\middle|\, i = 0, 1, \ldots, \frac{q+1}{3} \right\} \\ &\cup \left\{ (i-1, 3i+1) \,\middle|\, i = \frac{q+4}{3}, \ldots, m-2 \right\} \\ &\cup \left\{ (i+m-1, -3i-1) \,\middle|\, i = 0, 1, \ldots, \frac{q+1}{3} \right\} \\ &\cup \left\{ (i+m-2, -3i-2) \,\middle|\, i = \frac{q+4}{3}, \ldots, m-2 \right\}. \end{aligned}$$

This chessboard will contain two empty columns, the column $m-2$ and $m+1$. The rows $0, 1, 2, \ldots, m-3$ will each contain one white bishop. The rows $m-1, m, \ldots, q-3$ will each contain one black bishop. Hence, four rows will be empty: the rows $m-2$, $q-2$, $q-1$ and q. It is easy to see that one can make B' to a complete set B of bishops by adjoining two bishops in such way that the empty rows and empty columns will remain empty.

We first move the bishop at the position $(0,0)$ to its complementary position (m,m). We also delete the bishop at the position $(m-1, -1)$. Two new empty rows and two new empty columns will appear: the rows 0 and $m-1$ and columns 0 and -1. By completing with free lines we obtain maximal partial spreads. The deleted bishop will not be replaceable and thus the sizes of the maximal partial spreads will be

$$\frac{q^2+1}{2}+n \quad \text{for } n=12, 13, 14, \ldots, 18.$$

Next, we move the bishop at the position $(q-3, m+4)$ to its complementary position and complete with free lines. The size of the maximal partial spread will be

$$\frac{q^2+1}{2} + n \quad \text{for } n = 15, 16, 17, \ldots, 27.$$

We continue by moving the obtained set of bishops four steps down the chessboard. By completing with free lines we get maximal partial spreads of the sizes

$$\frac{q^2+1}{2} + n \quad \text{for } 28 \leqslant n \leqslant 47 \quad \text{and} \quad n = 25 \text{ and } 50.$$

We start again with the above set of bishops B. We delete the bishop at the position $(m-1, -1)$. If we complete with free lines in the usual way, the deleted bishop will not be replaceable. Hence, the number of lines of the maximal partial spreads obtained, will be

$$\frac{q^2+1}{2} + n \quad \text{for } n = 9, 11, 13.$$

Finally, we consider the set of bishops B again. We move the bishop at the position $(m-1, -1)$ to its complementary position and complete the partial spread with free lines. The sizes of the maximal partial spreads will be

$$\frac{q^2+1}{2} + n \quad \text{for } n = 8, 9, 10.$$

5.3. *The interval* $(q^2+1)/2 + 48 \leqslant n \leqslant (5q^2+4q-1)/8$

Let $m = (q+1)/2$. Assume that $2a \equiv 1 \pmod{m}$. Consider the following set of bishops on a chessboard of size $(m \times m)$:

$$\begin{aligned} B_0 = &\left\{ (i, 3i) \;\middle|\; i = 0, 1, \ldots, \frac{m}{3} - 1 \right\} \\ &\cup \left\{ \left(i + \frac{m}{3} - 1, 3i + 1\right) \;\middle|\; i = 0, 1, \ldots, \frac{m}{3} - 1 \right\} \\ &\cup \left\{ \left(i + \frac{2m}{3} - 2, 3i + 2\right) \;\middle|\; i = 0, 1, \ldots, \frac{m}{3} - 1 \right\} \\ &\cup \left\{ \left(-\frac{m}{3} - 2, \frac{m}{3} - 2\right), \left(\frac{-am}{3} - 5, \frac{am}{3} - 3\right) \right\}. \end{aligned}$$

The union of the set of bishops B_0 and the set of bishops obtained by moving all bishops above m steps in the y-direction will constitute a complete set B of bishops:

$$(x, y) \in B \quad \text{if and only if} \quad x, y \in B_0 \text{ or } (x, y + m \,(\text{mod}\,(q+1))) \in B_0.$$

If a white (black) bishop is moved m steps in the y-direction it will be transformed into a black (white) bishop. Hence, the rows $0, 1, 2, \ldots, m-5$ will each contain at least one white and one black bishop. The remaining $m+4$ rows will be empty. Four columns are empty. These are the columns $m-1$, $m-4$, $q-3$ and -1.

Using this set of bishops B and the set of bishops B^* obtained from B by deleting the bishop at the position $(m-5, q-6)$, it is a trivial exercise to construct maximal partial spreads of the sizes prescribed: we simply move all black bishops an even number of steps up the chessboard, and then, as in the last subsection of Section 3, we delete a bishop in a row already occupied by another bishop.

6. The case $q+1 \equiv 2$ or $10 \pmod{12}$

Let $m = (q+1)/2$. Let p be any integer such that p, $p-1$ and $p+1$ are relatively prime to m. In this section the following solutions to modular n-queen problem will be used:

$$Q_p = \{(m-1-x, px \pmod m) \mid x = 0, 1, 2, \ldots, m-1\}$$

and

$$Q_p^{\text{opp}} = \{(m-1-px \pmod m, x) \mid x = 0, 1, 2, \ldots, m-1\}.$$

B_p is defined in the following way:

$$(x, y) \in B_p \quad \text{if and only if} \quad (x, y) \in Q_p \quad \text{or} \quad (x, y + m \pmod{q+1}) \in Q_p.$$

We note that B_p is a complete set of bishops. B_p^{opp} will be defined in the same way.

We will use the following proposition.

Proposition 6.1. *If $m \equiv 1$ or $5 \pmod 6$ then for any integer d in the interval $1 \leqslant d < m$, there is a solution of the modular m-queen problem with one of the queens at the position $(m-1, 0)$ and another queen at the position $(m-1-e, d)$, where $1 \leqslant e < m/2$.*

Proof. Take any solution of the modular m-queen problem (e.g. the solution Q_2 above). Assume that the queen in column number 0 is placed at the position $(a, 0)$, and the queen in column number d is placed at the position (b, d).

If $a < b$ and $b - a < m/2$, then we move all queens in the following way: a queen at the position (i, j) is moved to the position $((m-1) - (i-a) \pmod m, j)$. This will be a solution with one queen at the position $(m-1, 0)$ and one queen at the position $(m-1-(b-a) \pmod m, d)$.

The other cases of the proposition can be proved with similar arguments. □

6.1. The interval $(q^2+1)/2 + 1 \leqslant n \leqslant (q^2+1)/2 + 5$

Consider the complete set of bishops B_3. We perform the following four operations on B_3:

(a) We move all black bishops $m-1$ step up the chessboard.
(b) We move the white bishops $m-1$ step up the chessboard and move the bishop at position $(2m-2, 0)$ to its complementary position.

(c) We move all black bishops $m-3$ step up the chessboard and delete the bishop at the position $(m-1, 0)$.
and
(d) As in (c) but we also delete the bishop at the position $(m-2, 3)$.

By using these operations and completing with free lines we can obtain maximal partial spreads of the sizes indicated above.

6.2. *The interval* $(q^2+1)/2+6 \leqslant n \leqslant (q^2+1)/2+(1+2+\cdots+(m-1)/2)+1$

In this case we consider the two complete sets of bishops B_2 and B_3.

Let b_1 be the bishop at the position $(m-1, 0)$. In the set of bishops B_3 we will let b_2 denote the bishop at the position $(m-1-x, 3x)$. In the set of bishops B_2 we will let b_2 denote the bishop at the position $(m-1-2x, 4x)$. (B_3 and B_2 will never be used in the same construction.) We note that both b_1 and b_2 are white bishops. Let b_3 be the bishop in the position $(m-1, m)$.

We will consider the following operations:

(a) For a suitable integer x, delete the bishop b_2 from the chessboard.
(b) Delete the bishop b_3 from the chessboard.
(c) Move the bishop b_1 to its complementary position.
(d) Move the bishop b_3 to its complementary position.
(e) Move all black bishops $2k$ steps up the chessboard.
(f) Switch bidiagonals for b_1 and b_2, i.e. b_1 is moved along its diagonal until it reaches the bidiagonal of the bishop b_2, and similar for b_2.

By combining these operations, or in fact some of them, we will obtain maximal partial spreads of the prescribed sizes.

6.3. *The interval* $(q^2+1)/2+(1+2+\cdots+(m-1)/2)+2 \leqslant n \leqslant (5q^2+4q-1)/8$

Let Q be any solution of the modular m-queen problem satisfying the condition of the Proposition 6.1, i.e. there is one queen (or bishop) b_1 at the position $(m-1, 0)$ and another queen (or bishop) b_2 at the position $(m-1-e, d)$. Let B be defined as the following complete set of bishops:

$$(x, y) \in B \quad \text{if and only if} \quad (x, y) \in Q \quad \text{or} \quad (x, y+m \,(\bmod\, q+1)) \in Q.$$

We will consider the following operations on this set B of bishops:

(a) We move all black bishops $2k$, $2k < m/2$, step up the chessboard.
(b) We move all white bishops $2k$, $2k < m/2$, step up the chessboard.
(c) We move the bishop b_1 to its complementary position.
(d) We delete the bishop b_1 from the chessboard.
(e) We delete the bishops b_2 from the chessboard.

We combine the above operations and complete with free lines. It is an easy (but most boring) exercise to verify that there are maximal partial spreads of the sizes promised above, with one exception: the maximal partial spreads of the sizes

$$\frac{q^2+1}{2} + (1+2+\cdots+(m-2k-1)) + 1, \quad 2k \leqslant m-5.$$

We obtain maximal partial spreads of these sizes by moving the black bishops of the complete set of bishops B_2^{opp}, $2k$ steps up the chessboard and deleting the two bishops at the positions $(m-1,0)$ and $(m-5,2)$.

7. The case $q+1 \equiv 4$ or $20 \pmod{24}$

The proof of the theorem in this case can be performed exactly in the same way as the proof of the theorem in the cases $q+1 \equiv 8$ or $16 \pmod{24}$ in [11].

8. Uncited references

[8,15]

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